

# INTRODUCTION TO PROBABILITY

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# INTRODUCTION TO PROBABILITY

Many questions in statistics involve uncertainty:

- Will a patient respond to a new treatment?
- What is the chance that a diagnostic test gives the correct result?
- How likely is a particular genetic trait to occur?

Probability gives us a way to describe and quantify uncertainty.

## RANDOM EXPERIMENTS

A **random experiment** is an action or process in which we know the possible outcomes of an event but are not known until the outcomes of the event have occurred.

### Examples:

- Flip a fair coin, what are the possible outcomes ?
- Roll a fair die, what is the chance of getting 1 when rolling a die?
- Draw one card from a standard deck of cards, what is the chance of drawing a heart?

## PROBABILITY

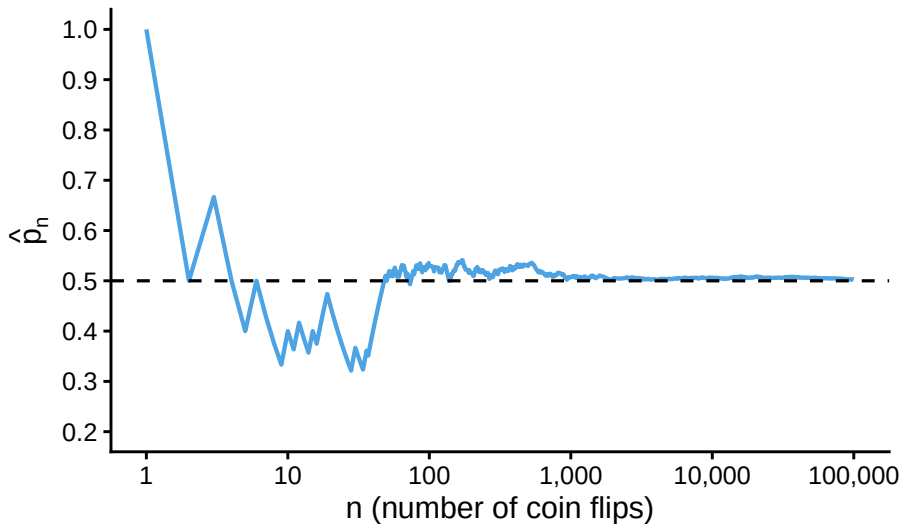
The **probability**  $p$  of an outcome describes how often we expect that outcome to occur when a random process is repeated many times (*in the long run*)

$$p = \frac{\text{number of times the outcome would occur}}{\text{total number of repetitions}}$$

**Example:** If a fair coin is flipped an infinite number of times, heads would be obtained 50% of the time

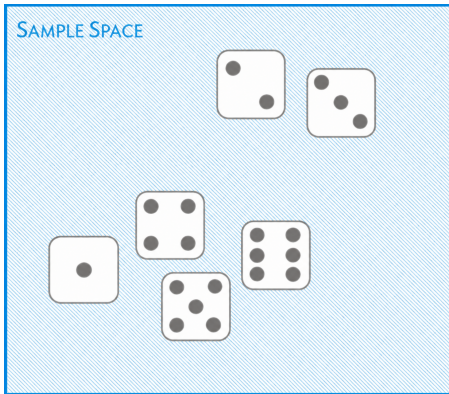
## LAW OF LARGE NUMBERS

As more observations are collected, the proportion  $\hat{p}_n$  of occurrences with a particular outcome converges to the probability  $p$  of that outcome



## SAMPLE SPACE

The set of all possible outcomes is called the **sample space**, denoted  $S$ .



Rolling a die produces a value in the set  $S = \{1, 2, 3, 4, 5, 6\}$ .

## GUIDED PRACTICE

For each of the following random processes, identify the sample space

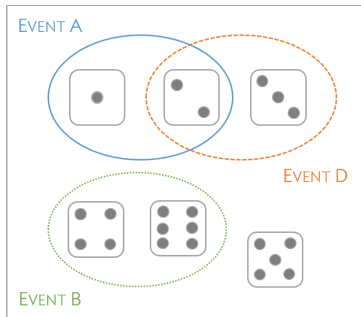
- Draw one card from a standard deck, record its suit
- Flip a coin twice
- Roll two dice and record the sum

## EVENTS

An **event** is a subset of outcomes from the sample space ( $S$ ). Any capital letter can be used to denote such event.

Possible events (*but not limited to*) for rolling a fair die include

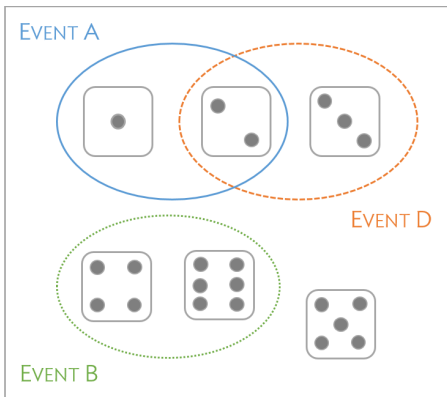
$A = \{1, 2\}$ ,  $B = \{4, 6\}$ ,  $D = \{2, 3\}$ .



Other commonly used events are “all even numbers”,  $C = \{2, 4, 6\}$

## DISJOINT/MUTUALLY EXCLUSIVE EVENTS

Two events or outcomes are called **disjoint** or **mutually exclusive** if they cannot both happen at the same time



Here, events *A* and *B* are disjoint

## ADDITION RULE FOR DISJOINT EVENTS

If  $A_1$  and  $A_2$  represent two **disjoint** outcomes, then the probability that either one of them occurs is given by

$$P(A_1 \text{ or } A_2) = P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

The symbol “ $\cup$ ” is called a **union** and is used to represent “or”.

## AXIOMS OF PROBABILITY

For each event  $A$  in the sample space  $S$  there exists a value  $P(A)$ , referred to as **the probability of  $A$** , which satisfied the following

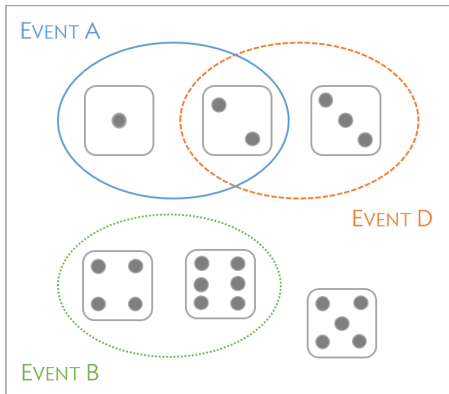
- $0 \leq P(A) \leq 1$
- $P(S) = 1$
- If  $A_1, A_2, \dots, A_n$  is a sequence of pairwise **disjoint** events then

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{i=1}^n P(A_i)$$

All theory around probability are based on these axioms

Consider rolling a fair die, what is the probability of rolling  $\{1, 2, 4, 6\}$ ?

Intuitively, rolling  $\{1, 2, 4, 6\}$  is  $\frac{4}{6} = \frac{2}{3}$



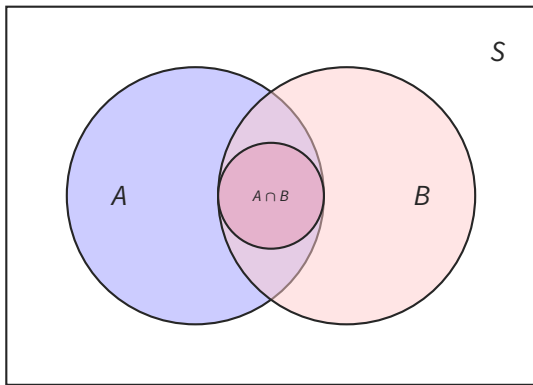
Using the addition rule, since  $A$  and  $B$  are disjoint then

$$P(A \cup B) = P(A) + P(B) = \frac{2}{6} + \frac{2}{6} = \frac{2}{3}$$

## GENERAL ADDITION RULE

In the case that  $A$  and  $B$  are not mutually exclusive, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



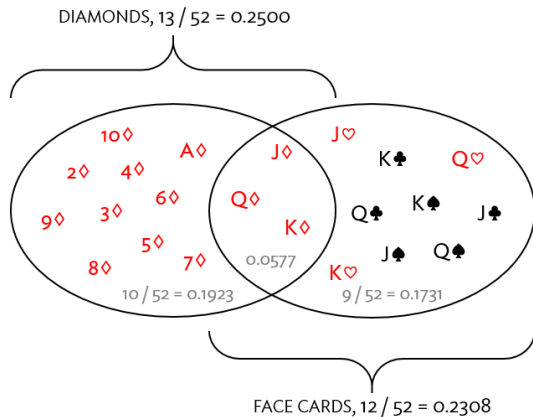
The symbol  $\cap$  denotes the “intersection” of two events

Suppose that we are interested in the probability of drawing a diamond or a face card out of a standard 52-card deck

|    |    |    |    |    |    |    |    |     |    |    |    |    |
|----|----|----|----|----|----|----|----|-----|----|----|----|----|
| 2♣ | 3♣ | 4♣ | 5♣ | 6♣ | 7♣ | 8♣ | 9♣ | 10♣ | J♣ | Q♣ | K♣ | A♣ |
| 2♦ | 3♦ | 4♦ | 5♦ | 6♦ | 7♦ | 8♦ | 9♦ | 10♦ | J♦ | Q♦ | K♦ | A♦ |
| 2♥ | 3♥ | 4♥ | 5♥ | 6♥ | 7♥ | 8♥ | 9♥ | 10♥ | J♥ | Q♥ | K♥ | A♥ |
| 2♠ | 3♠ | 4♠ | 5♠ | 6♠ | 7♠ | 8♠ | 9♠ | 10♠ | J♠ | Q♠ | K♠ | A♠ |

Are the events “drawing a diamond” and “drawing a face” card disjoint ?

To correct the double counting of the three cards that are in both events, subtract the probability that both events occur

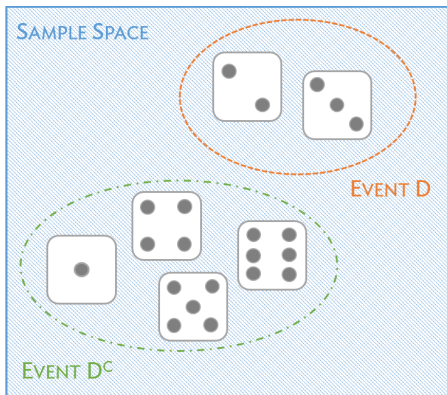


## GUIDED PRACTICE

Suppose researchers are studying symptoms among patients with a particular infection. They find that 60% of patients have a fever, 45% have a cough, and 25% have both a fever and a cough. If one patient is selected at random, what is the probability that the patient has a fever or a cough?

## COMPLEMENT OF AN EVENT

Let  $D = \{2, 3\}$  represent the event that the outcomes of a die roll is 2 or 3.



The **complement** of  $D$  represents all outcomes in the sample space that **are not** in  $D$ . That is  $\{1, 4, 5, 6\}$

The **complement** of an event  $A$  is denoted by  $A^c$ .

Since  $A$  and  $A^c$  are **disjoint**, we have  $P(A \cup A^c) = P(A) + P(A^c)$

If every possible outcome not in  $A$  is in  $A^c$ , then

$$P(S) = P(A \cup A^c) = P(A) + P(A^c) = 1$$

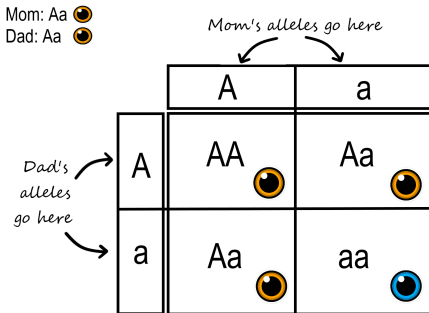
An event and its complement are mathematically related:

$$P(A) + P(A^c) = 1 \iff P(A) = 1 - P(A^c)$$

Similarly,  $P(A^c) = 1 - P(A)$

## GUIDED PRACTICE

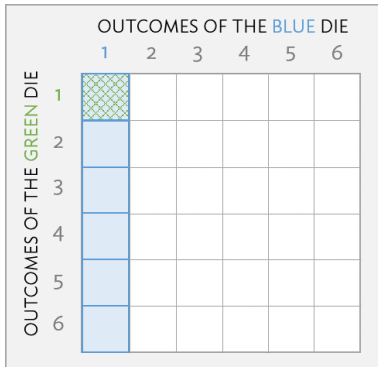
Two heterozygous parents have genotypes  $Aa \times Aa$ . What is the probability that their child inherits at least one dominant allele  $A$ ?



## INDEPENDENT EVENTS

Two events  $A$  and  $B$  are **independent** if the probability that both  $A$  and  $B$  occur equal the product of their separate probabilities

$$P(A \cap B) = P(A)P(B)$$



Two events are **independent** if knowing the outcome of one provides no useful information about the outcome of the other

# PROBABILITY DISTRIBUTIONS

## Definition

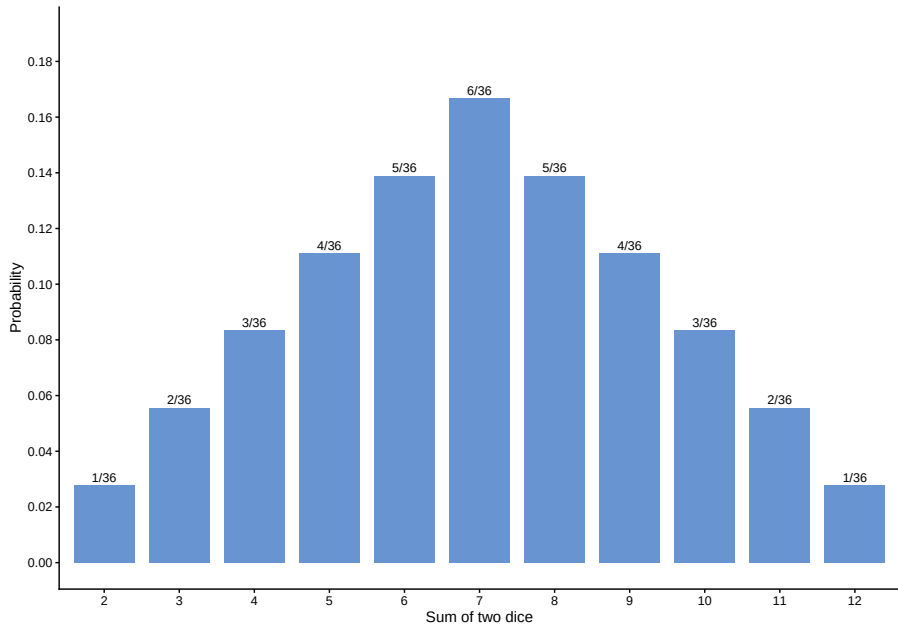
A probability distribution is a list of possible outcomes and their associated probabilities that satisfies three rules:

1. The outcomes listed must be disjoint
2. Each probability must be between 0 and 1
3. The probabilities must total to 1

## PROBABILITY DISTRIBUTION: SUM OF TWO DICE



| Dice sum    | 2              | 3              | 4              | 5              | 6              | 7              | 8              | 9              | 10             | 11             | 12             |
|-------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Probability | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |



## TYPES OF PROBABILITY DISTRIBUTIONS

Probability distribution for event that take on a **finite number of possible outcomes**, such as the sum two dice rolls, are referred to as **discrete probability distributions**

Probability distributions for events that can take on an **infinite number of possible values** within an interval, such as a person's height or the time required to complete a task, are referred to as **continuous probability distributions**.